# The Sum-Capacity of different $K$-user Cognitive Interference Channels in Strong Interference

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***Abstract*—This work considers different $K$-user extensions of the two-user cognitive interference channel model. The models differ by the cognitive abilities of the transmitters. In particular, the *primary message sharing* model, in which only one user is cognitive and knows all messages, and the *cumulative message sharing* model, in which a user knows the messages of all users with lesser index, are analyzed.**

**The central contribution is the characterization of the sum-capacity of both models under a *strong interference* condition, which amounts to having one receiver in the network that can decode all transmitted signals without loss of optimality. The sum-capacity is evaluated for the Gaussian noise channel, as well as the conditions on the channel gains that grant strong interference.**

## I. Introduction

Cognitive Radio (CR) has emerged as a promising technology to help alleviate spectrum scarcity. Wireless networks, in which certain nodes are equipped with CR technology to allow them to make more intelligent use of the spectrum, are termed *cognitive networks*. In cognitive networks, *secondary* users equipped with CR technology are assumed to coexist with licensed *primary users* (without CR capabilities). Secondary users exploit their knowledge of the network state, which they acquire thanks to their CR abilities, to communicate while controlling the performance degradation to the existing primary users. Cognitive network operation may be categorized as underlay, overlay and interweave [1]. This work focuses on the information theoretic overlay model, in which cognitive users a priori know the messages on the primary users.

**Past Work.** The information theoretic model for overlay two-user CR was introduced in [2]. To the best of our knowledge, the state-of-the-art on general inner and outer bounds, and on exact and approximate capacity results, is in [3], [4]. In particular, our work aims to extend the two-user strong interference capacity region result of [5] to multiple users. Due to the complexity of characterizing the capacity region of a general $K$-user channel, as a first step we investigate the sum-capacity of multiuser cognitive channels under strong interference. Limited results are available for CR channels with more than two users, due to the complexity of the problem and the many different message sharing possibilities.

In [6], [7] the authors considered a channel of two primary users and one cognitive user and provided the capacity region under strong interference, where the channel reduces to a compound MAC. In [8] a channel model that consists of one primary user and $K-1$ parallel cognitive users was considered and the capacity under very strong interference was derived by using lattice codes. In [9] the authors classify CR channels with three users as *primary message sharing* (PMS), *cumulative message sharing* (CMS), or *cognitive only message sharing* (CoMS). In PMS, one cognitive user knows the messages of the other two primary users; in CMS, the channel consists of one primary user, with index 1, and two cognitive users where each cognitive user knows the message of the users with lesser index; in CoMS, the channel consists of one primary user and two cognitive users that knows the primary's messages. In [10] we derived a general capacity upper bound for the CMS model and showed that the symmetric (all direct links have the same strength and all cross links have the same strength) sum-capacity in Gaussian noise can be achieved to within a constant gap. In this paper we consider a classification similar to [9] but for arbitrary number of users by focusing in particular on PMS and CMS under strong interference.

**Contributions.** The main contributions of this work are as follows. Result 1: a sum-rate outer bound valid for any number of users under a certain *strong interference condition*, which amount to having one receiver that can decode all transmitted signals without loss of optimality. The bound takes the same form for both PMS and CMS models, but over different sets of input distributions. The bound does not contain auxiliary random variables and is therefore computable for many channels of interest, including the Gaussian channel. The bound is not the classical compound MAC result of similar strong interference capacity results. Result 2: We present coding schemes that achieve the sum-rate outer bound for both PMS and CMS in strong interference. Result 3: For the Gaussian noise channel, we explicitly characterize the set of channel gains satisfying the strong interference condition and compare our results with [5] (for the 2-user case) and [6], [7] (for the 3-user case). Since we only focus on sum-capacity, our outer bound holds under more relaxed conditions than [6], [7]. Moreover, our approach extends beyond the 3-user Gaussian case to any number of users and to any memoryless channel.

**Paper Organization.** Section II describes the channel model. Section III contains our sum-rate outer bound for arbitrary number of users under strong interference. Section IV shows the achievability of the sum-rate outer bound. In Section V we evaluate the sum-capacity and the strong interference condition in Gaussian noise. Section VI concludes the paper. Proofs may be found in Appendix.

## II. CHANNELMODEL

We use the following notation convention: $[n_1 : n_2]$ is the set of integers from $n_1$ to $n_2 \geq n_1$; $Y^j$ is a vector of length $j$ with components $(Y_1, \ldots, Y_j)$; $\mathbf{I}_j$ is the identity matrix of dimension $j$; for an index set $\mathcal{A} \subseteq [1 : K]$ we let $X_{\mathcal{A}} = \{X_j, j \in \mathcal{A}\}$ or $R_{\mathcal{A}} = \sum_{\in \mathcal{A}} R_j$ (which one is usually clear form the context).

The general memoryless $K$-user cognitive interference channel ($K$-CIFC) consists of $K$ source-destination pairs sharing the same physical channel and is formally defined by channel inputs $X_i \in \mathcal{X}_i$, channel outputs $Y_i \in \mathcal{Y}_i$, $i \in [1 : K]$, and a memoryless channel $\mathbb{P}(Y_1, \ldots, Y_K | X_1, \ldots, X_K)$. A code with non-negative rate vector $(R_1, \ldots, R_K)$ and block-length $N$ is defined by: messages $W_i$, uniformly distributed over $[1 : 2^{NR_i}]$ and independent of all other random variables, encoding functions $X_i^N\big(W_{\mathcal{M}_i}\big)$ with $\mathcal{M}_i \subseteq [1 : K]$, and decoding functions $\widehat{W}_i\big(Y_i^N\big)$, for $i \in [1 : K]$. The capacity region is the set of all rate tuples $(R_1, \ldots, R_K)$ for which there exist a sequence of codes indexed by the block-length $N$ such that $P_e^{(N)} := \max_{i \in [1:K]} \mathbb{P}[\widehat{W}_i \neq W_i] \to 0$ as $N \to +\infty$.

We focus on two message assignments $(\mathcal{M}_1, \ldots, \mathcal{M}_K)$:

1) PMS: $\mathcal{M}_i = \{i\}$, $i \in [1 : K-1]$, and $\mathcal{M}_K = [1 : K]$,
2) CMS: $\mathcal{M}_i = [1 : i]$, $i \in [1 : K]$.

PMS and CMS are shown in Fig. 1(a) and Fig. 1(b), respectively, for the case of $K = 4$ users.

## III. OUTER BOUND

Our first result is a sum-capacity outer bound under a set of strong interference conditions:

**Theorem 1.** *For the $K$-CIFC-PMS and the $K$-CIFC-CMS satisfying the following condition $\forall j \in [2 : K]$*

$$I(X_{[j:K]}; Y_j | X_{[1:j-1]}) \leq I(X_{[j:K]}; Y_{j-1} | X_{[1:j-1]}), \quad (1)$$

*the sum-capacity is upper bounded by*

$$\sum_{j=1}^{K} R_j \leq I(X_{[1:K]}; Y_1), \quad (2)$$

*for all input distributions $P_{X_1,\ldots,X_K}$ that factor as*

1) *PMS:* $\prod_{j=1}^{K-1} P_{X_j} \; P_{X_K|X_1,\ldots,X_{K-1}}$,
2) *CMS:* $P_{X_1,\ldots,X_K}$.

*Proof:* The proof can be found in the Appendix. ■

Remarks:

1) The condition in (1) intuitively says that for all $j \in [2 : K]$, given that the signals $(X_1, \ldots, X_{j-1})$ have been removed, receiver $j$ can decode the remaining signals $(X_j, \ldots, X_K)$ at a lower rate than receiver $j-1$, which somehow implies that receiver $j-1$ can 'better decode' signal $X_j$ than the indented receiver $j$. Of all the receivers, receiver 1 is the 'most powerful' and the sum-capacity in (2) can be interpreted as 'joint decoding' of all transmit signals at receiver 1.
2) For $K = 2$ the PMS and the CMS models coincide and the condition in (1) reduces to [5, Eq. (93)], that is, $I(X_2; Y_2 | X_1) \leq I(X_2; Y_1 | X_1)$ for all $P_{X_1,X_2}$.
3) In [10] we derived the following sum-capacity upper bound for CMS case without any restriction

$$\sum_{j=1}^{K} R_j \leq \sum_{j=1}^{K} I(Y_j; X_{[j:K]} | X_{[1:j-1]}, Y_{[1:j-1]}). \quad (3)$$

We notice that (3) and Theorem 1 coincide for channels that satisfy the following *degradedness* condition

$$X_{[j:K]} \to Y_{j-1} \to Y_j \text{ given } X_{[1:j-1]}, \quad (4a)$$
$$\forall j \in [2 : K] \text{ and for all possible } P_{X_1,\ldots,X_K}. \quad (4b)$$

For the 3-user Gaussian channel, we shall see that the condition in (1) is less restrictive than (4).

## IV. INNER BOUND

### A. Achievable Scheme for $K$-CIFC-CMS

With CMS, message $W_1$ is known to all users. Thus all users may cooperate in sending message $W_1$ to receiver 1. In order to achieve the sum-outer bound, all users *beam form* to receiver 1 as in a MISO channel to achieve

$$R_1 = I(X_{[1:K]}; Y_1),$$
$$R_2 = \ldots R_K = 0,$$

for some $P_{X_1,\ldots,X_K}$. Hence, when the condition in (1) is satisfied for all input distributions, the sum-capacity of the $K$-CIFC-CMS is given by (2).

### B. Achievable Scheme for $K$-CIFC-PMS

With PMS, messages $W_1$ through $W_{K-1}$ are known at transmitter $K$. Here we propose a simple achievable scheme where users 1 through $K-1$ use independent i.i.d. coding (like in point-to-point channels) and user $K$ superposes its own message to the codewords generated by the other users. All destinations are required to decode all messages, where non-intended messages are decoded non-uniquely [11]. The achievable region is therefore the intersection of $K$ multiple access channels (with the difference that $X_K$ can be correlated to all other mutually independent inputs) given by

$$R_{\mathcal{S}} + R_K \leq \min_{j \in [1:K]} I(X_{\mathcal{S}}; Y_j | X_{\mathcal{S}^c}),$$

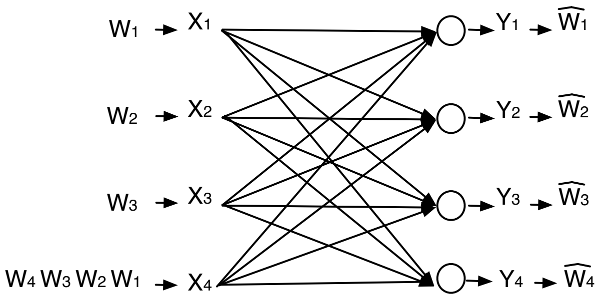
for all $\mathcal{S} \subseteq [1 : K-1] \backslash \emptyset$ and

$$R_K \leq I(X_K; Y_K | X_{[1:K-1]}).$$

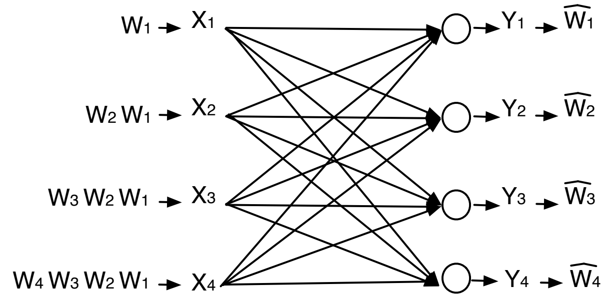
The achievable sum-rate is therefore obtained by $\mathcal{S} = [1 : K-1]$. To meet the sum-rate outer bound in (2) we need to impose the extra condition that

$$I(X_{[1:K]}; Y_1) \leq I(X_{[1:K]}; Y_j), \; j \in [2 : K], \quad (5)$$

for the distribution that attains the largest value in the sum-rate upper bound. Hence, when in addition to the condition in (1) being satisfied for the set of input distributions with the prescribed factorization, also the condition in (5) is satisfied, the sum-capacity of $K$-CIFC-PMS is given by (2).




(a) $K$-CIFC-PMS with $K = 4$.




(b) $K$-CIFC-CMS with $K = 4$.

## V. The Gaussian Noise Case

The power-constrained complex-valued single-antenna $K$-user Gaussian noise channel is described by the following input/output relationship

$$Y_j = \sum_{k\in[1:K]} h_{jk} X_k + Z_j,\ j \in [1:K], \tag{6}$$

where the channel gains $h_{jk} \in \mathbb{C}$, $(i,j) \in [1:K]^2$, are constant and known to all terminals, the noises are without loss of generality zero mean, unit variance proper-complex Gaussian random variables (their correlation does not matter as the receivers do not cooperate), and the inputs are subject to power constraint $\mathbb{E}[|X_k|^2] \le 1$, $k \in [1:K]$.

### A. Outer bound

In the next subsections we aim to evaluate Theorem 1 for the channel in (6). To do so, we need to identify the channels for which the strong interference condition in (1) holds for all input distributions with the proper factorization depending on the message sharing mechanism. Next we argue that for the power-constrained Gaussian channel, the strong interference condition must be verified only for those input distributions that meet the power constraint with equality for each user. The idea is that all other distributions are suboptimal in the sense that one can find another distribution with provably better performance. The proof is by contradiction. Assume that there is an optimal input distribution for which user $k \in [1:K]$ uses $\mathbb{E}[|X_k|^2] = P_k \le 1$ with a user $k^*$ such that $P_{k^*} < 1$. Consider now a new communication scheme in which user $k^*$ sends $X_{k^*,\mathrm{new}} = X_{k^*} + X'_{k^*}$ where $X_{k^*}$ is the signal that was assumed optimal with power $P_{k^*} < 1$ and $X'_{k^*} \sim \mathcal{N}(0, 1-P_k)$ is independent of everything else and has rate

$$R'_{k^*} = \log\left(1 + \min_{j\in[1:K]} \frac{|h_{jk^*}|^2(1-P_{k^*})}{1 + (\sum_{\ell\in[1:K]} \sqrt{|h_{j\ell}|P_\ell})^2}\right) > 0$$

Now, the rate of the new message is such that $X'_{k^*}$ can be decoded by all users by treating the signals assumed optimal as noise (no matter what their correlation structure is); after that, $X'_{k^*}$ is removed for the received signal and the system is equivalent to the one assumed optimal. Now, since the rate of user $k^*$ can be increased by $R'_{k^*} > 0$ we reached a contradiction. This shows that all users must use their full power. Therefore, for the power constrained Gaussian channel, one can repeat the same steps of the converse by considering only those input distributions that meet the power constraint with equality for all users. This implies that the strong interference condition must be verified only by these distributions (rather than all possible input distributions).

### B. Sum-Capacity for the $K$-CIFC-CMS

The sum-capacity upper bound in (2), by the 'Gaussian maximizes entropy' theorem [11], yields

$$\begin{aligned}\sum_{k=1}^{K} R_k &\le \max_{\boldsymbol{\Sigma}_x} \log\left(1 + \mathbf{h}_1^H \boldsymbol{\Sigma}_x \mathbf{h}_1\right) \\ &= \log\Big(1 + \Big(\sum_{k\in[1:K]} |h_{1k}|\Big)^2\Big),\end{aligned} \tag{7}$$

since we can consider any input covariance matrix $\boldsymbol{\Sigma}_x$. The sum-capacity in (7) is valid under the condition in (1), which amounts to verifying

$$h(Y_j|X_{[1:j-1]}) \le h(Y_{j-1}|X_{[1:j-1]})\ j \in [2:K], \tag{8}$$

over all proper-complex Gaussian distribution that meet the power constraint with equality [5]–[7].

The sum-capacity is achieved by beam forming (see Section IV-A) with

$$X_k = \exp\{-\mathrm{j}\angle h_{1k}\}\ U,\ k \in [1:K],\ U \sim \mathcal{N}(0,1). \tag{9}$$

### C. Sum-Capacity for the $K$-CIFC-PMS

Consider an input covariance matrix:

$$\boldsymbol{\Sigma}_x = \begin{bmatrix} \mathbf{I}_{K-1} & \rho \\ \rho^H & 1 \end{bmatrix} : \|\rho\|^2 \le 1 \tag{10}$$

where $\rho \in \mathbb{C}^{K-1\times 1}$ is a vector of correlation coefficients. The sum-capacity upper bound in (2), by the 'Gaussian maximizes entropy' theorem [11], is maximized by a jointly Gaussian input with covariance (10). We therefore obtain the following sum-capacity upper bound, for $\mathbf{h}_1^H = [h_{11}, h_{12}, \ldots, h_{1K}]$,

$$\sum_{j=1}^{K} R_j \le \max_{\boldsymbol{\Sigma}_x \text{ in eq.(10)}} \log\left(1 + \mathbf{h}_1^H \boldsymbol{\Sigma}_x \mathbf{h}_1\right)$$
$$= \log\left(1 + \left(|h_{1K}| + \sqrt{\sum_{j\in[1:K-1]} |h_{1j}|^2}\right)^2\right) \quad (11)$$

attained by $\rho_j = \lambda h_{1j}^*$ for $\lambda : \|\rho\|^2 = 1$. This optimal choice of correlation coefficients implies $R_K = 0$. The sum-capacity in (11) is valid under condition (1), which amounts to verifying

$$h(Y_j|X_{[1:j-1]}) \le h(Y_{j-1}|X_{[1:j-1]}) \; j \in [2 : K], \quad (12)$$

for all proper-complex Gaussian distributions with covariance matrix as in (10) [5]–[7].

The sum-rate in (11) is achievable by (see also Section IV-B)

$$X_j = T_j \text{ i.i.d. } \mathcal{N}(0, 1), \; j \in [1 : K - 1], \quad (13a)$$
$$X_K = \sum_{j=1}^{K-1} T_j \rho_j : |\rho_j| \propto |h_{1j}|, \; j \in [1 : K - 1], \quad (13b)$$

under the condition in (5), that is,

$$\mathbf{h}_1^H \boldsymbol{\Sigma}_x \mathbf{h}_1 \le \mathbf{h}_j^H \boldsymbol{\Sigma}_x \mathbf{h}_j, \; \forall j \in [2 : K - 1], \quad (14)$$
$$\mathbf{h}_j^H := [h_{j1}, h_{j2}, \ldots, h_{jK}] \quad (15)$$

for the choice of correlation coefficients implied by (13). Since $R_K = 0$, the condition in (14) need not to hold for $j = K$ as receiver $K$ does not have anything to decode.

### D. The case $K = 2$

For the 2-user case CMS and PMS coincide. The sum-capacity is given by (7), i.e., $R_1 = \log(1 + (|h_{11}| + |h_{12}|)^2)$ and $R_2 = 0$, under the condition in (8) for $K = j = 2$, which is equivalent to

$$\log(1 + |h_{22}|^2) \le \log(1 + |h_{12}|^2) \iff |h_{22}|^2 \le |h_{12}|^2.$$

The achievability condition in (14) does not play a role for $K = 2$ because $R_2 = 0$.

Remark: The strong interference condition $|h_{22}|^2 \le |h_{12}|^2$ is equivalent to [5, eq.(87)]. However, the strong interference capacity region in [5, Theorem 5] also requires [5, eq.(88)]. This is the case since in order to determine the sum-capacity only less restrictive conditions are needed compared to the case where the whole capacity region must be characterized.

### E. The case $K = 3$ with CMS

For CMS and $K = 3$ we consider all jointly Gaussian inputs with covariance matrix given by

$$\text{Cov}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1^* & 1 & \rho_3 \\ \rho_2^* & \rho_3^* & 1 \end{bmatrix} : |\rho_i| \le 1, i = 1, 2, 3,$$
$$|\rho_3 - \rho_1\rho_2^*|^2 \le (1 - |\rho_1|^2)(1 - |\rho_2|^2).$$

For CMS the optimal sum-rate in (7) is obtained for $R_1 = \log(1 + (|h_{11}| + |h_{12}| + |h_{13}|)^2), R_2 = R_3 = 0$ by beamforming. The condition in (8) is: for $j = 3$, by proceeding similarly to the case $K = j = 2$ discussed previously in Section V-D, we have

$$|h_{33}|^2 \le |h_{23}|^2, \quad (16)$$

and for $j = 2$ we must find the channel gains that satisfy

$$\log(1 + \mathbf{h}_2^H \mathbf{S} \mathbf{h}_2) \le \log(1 + \mathbf{h}_1^H \mathbf{S} \mathbf{h}_1)$$
$$\mathbf{h}_2^* := \begin{bmatrix} h_{22} \\ h_{23} \end{bmatrix}, \mathbf{h}_1^* := \begin{bmatrix} h_{12} \\ h_{13} \end{bmatrix}, \mathbf{S} := \begin{bmatrix} 1 - |\rho_1|^2 & \rho_3 - \rho_1\rho_2^* \\ \rho_3^* - \rho_1^*\rho_2 & 1 - |\rho_2|^2 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} h_{22} \\ h_{23} \end{bmatrix} = \xi \begin{bmatrix} h_{12} \\ h_{13} \end{bmatrix} : |\xi| \le 1. \quad (17)$$

Remark: The condition in (17) corresponds to the 'degraded channel condition when conditioning on $X_1$' in (4). Given the message structure of CMS, there are so many coding possibilities at the transmitters that the channel conditions under which joint decoding of all messages at the least cognitive receiver is optimal only includes a form of 'degraded channel'. This suggests that for CMS and generic channel gains, other decoding strategies are sum-capacity optimal, see for example the symmetric sum-capacity result in [10]. Notice that here we did not ask for the conditions under which joint decoding of all messages at all receivers is optimal, i.e., when the channel reduces to compound MAC. If we were to ask for which channel gains joint decoding of all messages at a 'more cognitive receiver' than receiver 1 is optimal, we would generally find different conditions than the one in (17).

### F. The case $K = 3$ with PMS

For PMS and $K = 3$ we consider all jointly Gaussian inputs with covariance matrix given by

$$\text{Cov}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \rho_2 \\ 0 & 1 & \rho_3 \\ \rho_2^* & \rho_3^* & 1 \end{bmatrix} : |\rho_2|^2 + |\rho_3|^2 \le 1.$$

For PMS the optimal sum-rate is obtained when $R_1 + R_2 = \log(1 + (|h_{13}| + \sqrt{|h_{11}|^2 + |h_{12}|^2})^2), R_3 = 0$. The condition in (8) for $j = 3$ is as (16), while for $j = 2$ is

$$|h_{22}|^2 + |h_{23}|^2 + 2|h_{22}h_{23}^* - h_{12}h_{13}^*| \le |h_{12}|^2 + |h_{13}|^2 \quad (18)$$

which includes the 'degraded condition' in (17).

The condition for achievability in (14) evaluated for $j = 2$ imposes that the channel gains satisfy

$$\mathbf{h}_1^H \mathbf{S} \mathbf{h}_1 \le \mathbf{h}_2^H \mathbf{S} \mathbf{h}_2, \; \mathbf{h}_1^* := \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \end{bmatrix}, \mathbf{h}_2^* := \begin{bmatrix} h_{21} \\ h_{22} \\ h_{23} \end{bmatrix},$$
$$\mathbf{S} := \text{Cov}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \text{ with } \begin{matrix} \rho_2 = \frac{h_{11}^*}{\sqrt{|h_{11}|^2 + |h_{12}|^2}}, \\ \rho_3 = \frac{h_{12}^*}{\sqrt{|h_{11}|^2 + |h_{12}|^2}}. \end{matrix} \quad (19)$$

Remark: Interestingly, the condition in (18) is equivalent to [5, eq.(88)].

## G. Comparison with similar work

It is not straightforward to compare our sum-capacity results with prior work. In particular, in [7, Th. 4] the capacity region of the 3-user Gaussian CIFC with real-valued channel gains in strong interference with PMS was derived. Eight channel gain conditions are imposed [7, Th. 4] in order to derive the whole capacity region; in this work we consider complex-valued channel gains and our two channel gain conditions are subsets of those in [7, Th. 4] (i.e. two channel gain relations in [7, Th. 4] are equivalent to (16) and (18) in this work). The channel gain relationship imposed by the achievability scheme in (14) is however different from that in [7] since in the latter all receivers decode all messages as in a compound MAC; here we consider only the sum-capacity and hence we do not impose as restrictive decoding conditions for achievability as those in [7].

# VI. Conclusion

Little work exists on cognitive interference channels with more than 3 pairs of users. We make progress by considering a general $K$-user cognitive interference channel and derive its sum-capacity under a set of strong interference conditions. Our outer bound imposes fewer conditions than prior work for the case of $K = 3$ users. In addition, in contrast to most strong interference capacity results, our achievability scheme does not reduce to a compound multiple access channel. This in turn makes it challenging to compare the strong interference channel gain conditions with prior work for $K = 3$. We note that our outer bound is computable for any $K$ and that our strong interference sum-capacity result holds for any memoryless channel (and not just in Gaussian noise).

# Appendix

In order to obtain the sum-rate upper bound in (2) we first present the following Lemma, which is an extension to any $K$ of [5, Lemma 5] (for CMS) and [6, Lemma 1] (for PMS):

**Lemma 2.** *If per-letter condition in* (1) *is satisfied for all prescribed input distributions then*

$$I(X^N_{[j:K]}; Y^N_j | X^N_{[1:j-1]}) \le I(X^N_{[j:K]}; Y^N_{j-1} | X^N_{[1:j-1]}), \quad (20)$$

*for all input distributions* $P_{X_1^N,\ldots,X_K^N}$, $N \in \mathbb{N}$, *that factor as*

1) *PMS:* $\prod_{j=1}^{K-1} P_{X_j^N} \; P_{X_K^N|X_1^N,\ldots,X_{K-1}^N}$,
2) *CMS:* $P_{X_1^N,\ldots,X_K^N}$.

We are now ready to present the proof of Theorem 1: For $\epsilon_N > 0 : \epsilon_N \to 0$ as $N \to +\infty$, we have

$$N\sum_{j=1}^{K}(R_j - \epsilon_N) \overset{(a)}{\le} \sum_{j=1}^{K} I(W_j; Y_j^N)$$

$$\overset{(b)}{\le} \sum_{j=1}^{K} I(W_j; Y_j^N | W_{[1:j-1]}) \le \sum_{j=1}^{K} I(X_j^N; Y_j^N | X^N_{[1:j-1]})$$

$$\overset{(c)}{=} \sum_{j=1}^{K-1} I(X_j^N; Y_j^N | X^N_{[1:j-1]}) + I(X_K^N; Y_K^N | X^N_{[1:K-1]})$$

$$\overset{(d)}{\le} \sum_{j=1}^{K-1} I(X_j^N; Y_j^N | X^N_{[1:j-1]}) + I(X_K^N; Y_{K-1}^N | X^N_{[1:K-1]})$$

$$\overset{(e)}{=} \sum_{j=1}^{K-2} I(X_j^N; Y_j^N | X^N_{[1:j-1]}) + I(X^N_{[K-1:K]}; Y_{K-1}^N | X^N_{[1:K-2]})$$

$$\overset{(f)}{\le} \sum_{j=1}^{K-2} I(X_j^N; Y_j^N | X^N_{[1:j-1]}) + I(X^N_{[K-1:K]}; Y_{K-2}^N | X^N_{[1:K-2]})$$

$$\ldots \overset{(g)}{\le} I(X^N_{[1:K]}; Y_1^N) \overset{(h)}{\le} \sum_{t=1}^{N} I(X_{1t}, \ldots, X_{Kt}; Y_{1t})$$

$$\overset{(i)}{\le} NI(X_{[1:K]}; Y_1 | Q) \overset{(j)}{\le} NI(X_{[1:K]}; Y_1),$$

where: (a) follows from Fano's inequalities $H(W_j|Y_j^N) \le N\epsilon_N$, $\forall j \in [1:K]$, (c) from the independence of messages, the definition of encoding functions (for all $\mathcal{M}_j \subseteq [1:j]$, $j \in [1:K]$) and data processing inequality, (d), (f) and (g) from the condition in (1) for $j = K$, $j = K-1$, up to $j = 2$ and Lemma 2, (h) from chain rule of entropy, conditioning reduces entropy and memoryless property of the channel, (i) by introducing a time-sharing random variable $Q \sim \text{Unif}[1:N]$ and independent of everything else, and (j) by conditioning reduces entropy.


# Acknowledgment

The work of the authors was partially funded by NSF under award 1017436. The contents of this article are solely the responsibility of the authors and do not necessarily represent the official views of the NSF.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